Algebra : polynomials

Introduction

A polynomial is an **expression** which:

- consists of a sum of a **finite** number of terms
- has terms of the form $\mathbf{kx^n}$
  (**x** a variable, **k** a constant, **n** a positive integer)

Every polynomial in one variable (eg 'x') is equivalent to a polynomial with the form:

$$a_n x^n + a_{n-1} x^{n-1} + a_{n-1} x^{n-1} \ldots + a_2 x^2 + a_1 x^1 + a_0$$

Polynomials are often described by their **degree of order**. This is the highest index of the variable in the expression.

eg: containing x^5 order 5, containing x^7 order 7 etc.

These are **NOT** polynomials:

$$3x^2+x^{1/2}+x$$

second term has an index which is not an integer(whole number)

$$5x^{-2}+2x^{-3}+x^{-5}$$

indices of the variable contain integers which are not positive

examples of polynomials:

$$x^5+5x^2+2x+3$$

$$(x^7+4x^2)(3x-2)$$

$$x+2x^2-5x^3+x^4-2x^5+7x^6$$

Algebraic long division

If

$f(x)$ the numerator and **$d(x)$** the denominator are polynomials

and

the degree of $d(x)$ <= the degree of $f(x)$

and

$d(x)$ does not =0

then two unique polynomials **$q(x)$** the quotient and **$r(x)$** the remainder exist, so that:

$$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

Note - the degree of $r(x)$ < the degree of $d(x)$.

We say *that* $d(x)$ divides evenly into $f(x)$ when $r(x)=0$.

Example

$$\frac{5x^3 + x^2 - 3x + 2}{x^2 - 2x + 5}$$

$$\begin{array}{r|l} & \quad 5x + 11 \\ \hline x^2 - 2x + 5 & 5x^3 + x^2 - 3x + 2 \\ & -(5x^3 - 10x^2 + 25x) \\ & \qquad 11x^2 - 28x + 2 \\ & \qquad -(11x^2 - 22x + 55) \\ & \qquad\qquad -6x - 53 \end{array}$$

$$\begin{aligned} \frac{5x^3 + x^2 - 3x + 2}{x^2 - 2x + 5} &= 5x + 11 + \left(\frac{-6x - 53}{x^2 - 2x + 5}\right) \\ &= 5x + 11 - \left(\frac{6x + 53}{x^2 - 2x + 5}\right) \end{aligned}$$

The Remainder Theorem

If a polynomial f(x) is divided by (x-a), the remainder is f(a).

Example

Find the remainder when ($2x^3$+3x+x) is divided by (x+4).

$f(x) = 2x^3 + 3x^2 + x$ is divided by $(x+4)$

If a polynomial $f(x)$ is divided by $(x-a)$,
the remainder is $f(a)$.

$\Rightarrow \quad (x+4) = (x-a), \quad \therefore a = -4$

$$f(-4) = 2(-4)^3 + 3(-4)^2 + (-4)$$
$$= -128 + 48 - 4$$
$$= -84$$

the remainder is -84

The reader may wish to verify this answer by using algebraic division.

The Factor Theorem
(a special case of the Remainder Theorem)

$(x-a)$ is a factor of the polynomial $f(x)$ if $f(a) = 0$

Example

use the Factor Theorem to find factors of the function

$f(x) = x^3 + 3x^2 - x - 3$

choosing factors of the highest constant 3

1,-1,3,-3

$f(1) = (1)^3 + 3(1)^2 - (1) - 3$

$= 1 + 3 - 1 - 3 = 0$ $\quad \therefore \underline{(x-1) \text{ a factor}}$

$f(-1) = (-1)^3 + 3(-1)^2 - (-1) - 3$

$= -1 + 3 + 1 - 3 = 0$ $\quad \therefore \underline{(x+1) \text{ a factor}}$

$f(3) = (3)^3 + 3(3)^2 - (3) - 3$

$= 27 + 27 - 3 - 3 = 48$ $\quad \therefore \underline{(x-3) \text{ not a factor}}$

$f(-3) = (-3)^3 + 3(-3)^2 - (-3) - 3$

$= -27 + 27 + 3 - 3 = 0$ $\quad \therefore \underline{(x+3) \text{ a factor}}$

$\therefore \quad \underline{x^3 + 3x^2 - x - 3 = (x-1)(x+1)(x+3)}$

n.b. the sign change of the constant $f(5) \Rightarrow (x-5)$