Algebra : Functions

Introduction

To thoroughly understand the terms and symbols used in this section it is advised that you visit 'sets of numbers' first.

Mapping(or function)

This a '**notation**' for expressing a relation between two variables(say x and y).

Individual values of these variables are called **elements**

eg $x_1\ x_2\ x_3\ldots\quad y_1\ y_2\ y_3\ldots$

The first set of elements (x) is called **the domain** .

The second set of elements (y) is called **the range** .

A simple relation like $y = x^2$ can be more accurately expressed using the following format:

$$\{(x, y) : y = x^2, x, y \in \mathbb{R}\}$$

The last part relates to the fact that x and y are elements of the set of real numbers R(any positive or negative number, whole or otherwise, including zero)

One-One mapping

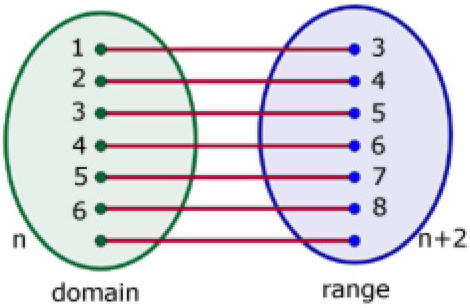
Here one element of the domain is associated with one and only one element of the range.

A property of one-one functions is that a on a graph a horizontal line will only cut the graph once.

Example

$$\{(x, y) : y = x + 2, x, y \in \mathbb{R}^+\}$$

R^+ the set of positive real numbers



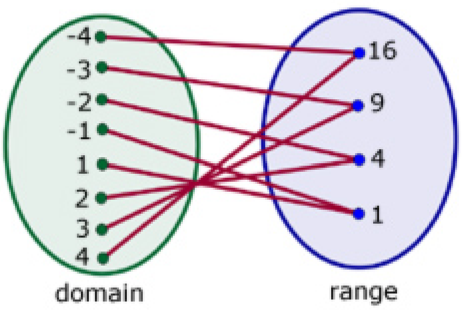
Many-One mapping

Here more than one element of the domain can be associated with one particular element of the range.

Example

$$\{(x, y) : y = x^2, x, y \in \mathbb{Z}, \quad -4 \leq x \leq 4, \quad x \neq 0\}$$

Z is the set of integers(positive & negative whole numbers not including zero)



Complete function notation is a variation on what has been used so far. It will be used from now on.

$$\{(x, y) : y = x^2,\ x, y \in \mathbb{R}\}$$

becomes $$\{f : x \mapsto x^2,\ x \in \mathbb{R}\}$$

Inverse Function **f^{-1}**

The **inverse function** is obtained by interchanging x and y in the function equation and then rearranging to make y the subject.

If f^{-1} exists then,

$$ff^{-1}(x) = f^{-1}f(x) = x$$

It is also a condition that the two functions be 'one to one'. That is that the domain of **f** is identical to the range of its inverse function **f^{-1}** .

When graphed, the function and its inverse are reflections either side of the line $\boldsymbol{y = x}$.

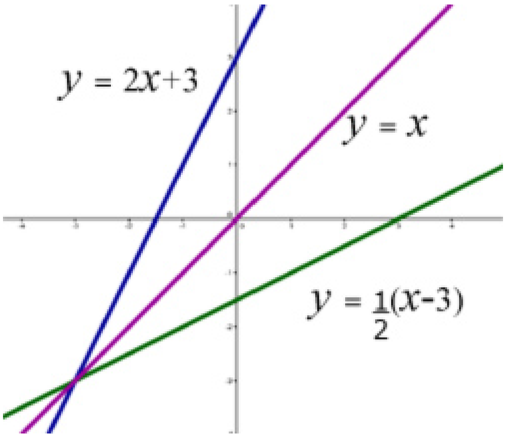
Example

Find the inverse of the function(below) and graph the function and its inverse on the same axes.

$$\{f : x \mapsto 2x + 3,\ x \in \mathbb{R}\}$$

$$\begin{aligned}
&\Rightarrow \quad y = 2x + 3 \\
&\text{interchanging } x \text{ and } y \\
&\Rightarrow \quad x = 2y + 3 \\
&\Rightarrow \quad x - 3 = 2y \\
&\Rightarrow \quad 2y = x - 3 \\
&\Rightarrow \quad y = \tfrac{1}{2}(x - 3)
\end{aligned}$$

$$\Rightarrow \quad \{f^{-1} : x \mapsto \tfrac{1}{2}(x - 3),\ x \in \mathbb{R}\}$$



Composite Functions

A **composite function** is formed when two functions f, g are combined.

However it must be emphasized that the order in which the composite function is determined is important.

$$f[g(x)] \neq g[f(x)]$$

The method for finding composite functions is:

find $g(x)$

find $f[g(x)]$

Example

For the two functions,

$$\{f : x \mapsto 2x - 1, \quad x \in \mathbb{R}\}$$
$$\{g : x \mapsto 3x + 2, \quad x \in \mathbb{R}\}$$

find the composite functions (i fg (ii gf

$$\{f : x \mapsto 2x - 1, \quad x \in \mathbb{R}\}$$
$$\{g : x \mapsto 3x + 2, \quad x \in \mathbb{R}\}$$

$$\begin{aligned} fg(x) &= f(3x+2) \\ &= 2(3x+2) - 1 \\ &= 6x + 4 - 1 \\ &= 6x + 3 \end{aligned}$$

$$\Rightarrow \underline{\{fg : x \mapsto,\ 6x + 3, \quad x \in \mathbb{R}\}}$$

$$\begin{aligned} gf(x) &= g(2x-1) \\ &= 3(2x-1) + 2 \\ &= 6x - 3 + 2 \\ &= 6x - 1 \end{aligned}$$

$$\Rightarrow \underline{\{gf : x \mapsto,\ 6x - 1, \quad x \in \mathbb{R}\}}$$

Exponential & Logarithmic Functions

Exponential functions have the general form:

$$\{f : x \mapsto a^x, \quad x \in \mathbb{R}\}$$

where 'a' is a positive constant

However there is a specific value of 'a' at (0.1) when the gradient is 1 . This value, **2.718...** or '**e**' is called the **exponential function**.

$$\{f : x \mapsto e^x, \quad x \in \mathbb{R}\}$$

The function(above) has one-one mapping. It therefore possesses an inverse. This inverse is the **logarithmic function**.

$$\begin{aligned}
& y = e^x \\
\Rightarrow \quad & \therefore \text{ the inverse is} \quad x = e^y \\
\Rightarrow \quad & \log_e x = \log_e e^y \\
\Rightarrow \quad & \log_e x = y \log_e e \\
\text{but} \quad & \log_e e = 1 \\
\therefore \quad & \log_e x = y \\
& y = \log_e x \\
\text{or} \quad & \underline{y = \ln x}
\end{aligned}$$