The Binomial Theorem

Introduction

This section of work is to do with the expansion of $(a+b)^n$ and $(1+x)^n$.

Pascal's Triangle and the Binomial Theorem gives us a way of expressing the expansion as a sum of ordered terms.

Pascal's Triangle

This is a method of predicting the coefficients of the binomial series.

Coefficients are the **constants**(1,2,3,4,5,6 etc.) that multiply each variable, or group of variables.

Consider $(a+b)^n$ variables a, b .

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

The first line represents the coefficients for n=0.

$$(a+b)^0 = 1$$

The second line represents the coefficients for n=1.

$$(a+b)^1 = a + b$$

The third line represents the coefficients for n=2.

$$(a+b)^2 = a^2 + 2ab + b^2$$

The sixth line represents the coefficients for n=5.

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

The Binomial Theorem builds on Pascal's Triangle in practical terms, since writing out triangles of numbers has its limits.

The General Binomial Expansion (n≥1)

This is a way of finding all the terms of the series, the coefficients and the powers of the variables.

The coefficients, represented by ^nC_r , are calculated using probability theory. For a deeper understanding you may wish to look at where ^nC_r comes from; but for now you must accept that:

$$^nC_r = \frac{n!}{(n-r)!r}$$

where 'n' is the power/index of the original expression
and 'r' is the number order of the term minus one

If n is a positive integer, then:

$$(a+b)^n = a^n + {^nC_1}a^{n-1}b + {^nC_2}a^{n-2}b^2 \ldots + {^nC_r}a^{n-r}b^r + \ldots b^n$$

Example #1

$$(a+b)^5 = a^5 + {^5C_1}a^4b \;+\; {^5C_2}a^3b^2 \;+\; {^5C_3}a^2b^3 \;+\; {^5C_4}ab^4 \;+\; b^5$$

$$^5C_1 = \frac{5!}{(5-1)!1!} = \frac{5.4.3.2.1}{(4.3.2.1)1} = 5$$

$$^5C_2 = \frac{5!}{(5-2)!2!} = \frac{5.4.3.2.1}{(3.2.1)2.1} = \frac{5.4.}{2.1} = \frac{20}{2} = 10$$

$$^5C_3 = \frac{5!}{(5-3)!3!} = \frac{5.4.3.2.1}{(2.1)3.2.1} = \frac{5.4.}{2.1} = \frac{20}{2} = 10$$

$$^5C_4 = \frac{5!}{(5-4)!4!} = \frac{5.4.3.2.1}{(1)4.3.2.1} = 5$$

$$(a+b)^5 = a^5 + 5a^4b \;+\; 10a^3b^2 \;+\; 10a^2b^3 \;+\; 5ab^4 \;+\; b^5$$

Example #2

write down $(3x-2)^5$ as a binomial expansion

using

$$(a+b)^5 = a^5 + {}^5C_1a^4b + {}^5C_2a^3b^2 + {}^5C_3a^2b^3 + {}^5C_4ab^4 + b^5$$

and the values of the coefficients from previous

let $a = 3x$, $b = -2$

$$(3x-2)^5 = (3x)^5 + {}^5C_1(3x)^4(-2) + {}^5C_2(3x)^3(-2)^2 + {}^5C_3(3x)^2(-2)^3 + {}^5C_4(3x)(-2)^4 + (-2)^5$$

$$(3x-2)^5 = 243x^5 + 5(81x^4)(-2) + 10(27x^3)(4) + 10(9x^2)(-8) + 5(3x)(16) + (-32)$$

$$(3x-2)^5 = 243x^5 - 810x^4 + 1080x^3 + 720x^2 + 240x - 32$$

It is suggested that the reader try making similar questions, working through the calculations and checking the answer ____ (max. value of n=8)

The Particular Binomial Expansion

This is for $(1+x)^n$, where n can take any value positive or negative, and **x is a fraction** ($-1<x<1$).

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

Example

Find the first 4 terms of the expression $(x+3)^{1/2}$.

$$\text{making } (x+3)^{1/2} \text{ in the form of } (1+x)^n$$

$$(x+3)^{1/2} = (3+x)^{1/2} = \left\{3\left(1+\frac{x}{3}\right)\right\}^{1/2}$$

$$= \sqrt{3}\left[1 + \frac{\left(\frac{1}{2}\right)\left(\frac{x}{3}\right)}{1} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{x}{3}\right)^2}{2.1} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{x}{3}\right)^3}{3.2.1}\right]$$

$$= \sqrt{3}\left[1 + \frac{x}{6} - \frac{1}{4}.\frac{1}{2}.\frac{x^2}{9} + \frac{3}{8}.\frac{1}{6}.\frac{x^3}{27} + \ldots\right]$$

$$= \sqrt{3}\left[1 + \frac{x}{6} - \frac{x^2}{72} + \frac{3x^3}{1296} + \ldots\right]$$

$$\underline{(x+3)^{1/2} = \sqrt{3} + \frac{\sqrt{3}x}{6} - \frac{\sqrt{3}x^2}{72} + \frac{\sqrt{3}x^3}{1296} \ldots}$$